

SURFACE DEFORMATION FOR CONSTANT STRESS DROP SLIP ZONE IN AN ELASTIC HALF-SPACE- ANTI-PLANE CASE

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Abstract

In this study, the static deformation of a fault system under constant stress drop is determined through the zero frequency approximation from a corresponding dynamic crack problem. The corresponding dynamic crack problem is formulated and solved by hypersingular boundary integral equation method (HBIEM). The relative displacement of the crack surface as well as the dynamic stress intensity factor of the crack tips can be calculated. Some numerical results are presented and checked by other available results, we also address some possible implication in geophysics and earthquake engineering.

INTRODUCTION

Detail understanding of the surface and internal deformations induced by a stress drop slip zone during an earthquake is valuable for geophysics and earthquake engineering. Actually, there are two approaches, namely, the stress drop method and the more common dislocation approach to obtain the surface deformation of a fault system. The dislocation approach models slip zones by approximation them as surfaces of prescribed displacement discontinuity. As pointed out by Wu *et al* (1991), a disadvantage of the dislocation approach is that it is purely kinematic: it relates surface deformation to slip at depth but does not directly yield information about the stresses that produced the slip. More specifically, when the distribution of slip is approximated by constant slip elements, the average stress change is unbounded, although it has an advantage due to its simplicity; therefore, in this study we adopt the stress drop method to obtain the surface deformation of a fault system.

To describe the general procedure briefly, firstly, the Green's function method is adopted in formulating the desired traction boundary integral equation. Then, the traction BIE can be solved through the discretization of the crack surface in which a special interpolation function is employed to satisfy the edge condition of the crack tips, and the boundary condition is satisfied by using the method of point collocation. A regularization technique is proposed to regularize the traction BIE. Based on the spectral integral form of the kernel function, the unbounded term can be isolated and extracted from the hypersingular kernel function by using the method of subtracted and added back in spectral domain, and hence, the required finite part can be obtained. Accordingly, the relative displacement of the crack surface as well as the dynamic stress intensity factor (DSIF) of the crack tips can be calculated, and they are essential in the evaluation of energy release rate and seismic moment during an earthquake. Some numerical results for the scattering problem of a crack subjected to plane

SH-waves, either in infinite or half-space, are implemented and the validity of the proposed method can be checked and verified by making against with other available results. Some possible implications in geophysics and earthquake engineering are also addressed.

PROBLEM DESCRIPTION

A stress-free crack of width $2a$ is embedded in a semi-infinite elastic medium. As shown in Fig. 1, two different right-handed Cartesian co-ordinate systems (x, y, z) and $(\mathbf{x}, y, \mathbf{h})$ with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_h)$, respectively. The crack Σ makes an angle d with the stress-free boundary of the half-space. The relation between the two co-ordinate systems can be written as

$$x = \mathbf{x} \cos d + \mathbf{h} \sin d, \quad z = d - \mathbf{x} \sin d + \mathbf{h} \cos d \quad (1)$$

where d ($d > a$) denotes the depth of the center of the crack from the boundary.

The material of the half-space is assumed to be isotropic, linear, homogeneous and perfectly elastic with Lamé constants l , m and mass density r . In this configuration, time-harmonic SH-waves are present and the complex time-factor $\exp(i\omega t)$ omitted, the governing equation of elastic motion is

$$\partial s_{xy} / \partial x + \partial s_{zy} / \partial z + r\omega^2 u_y = 0; \quad (x, z) \in V \quad (2)$$

where s_{xy} and s_{yz} are components of the stress tensor, u_y denotes the y -component of the particle displacement. The stress is related to the displacement by the constitutive relation,

$$s_{xy} = m \partial u_y / \partial x, \quad s_{yz} = m \partial u_y / \partial z \quad (3)$$

Elimination of the stress from equations (2) and (3) leads to

$$\partial^2 u_y / \partial x^2 + \partial^2 u_y / \partial z^2 + k_s^2 u_y = 0 \quad (4)$$

where k_s denotes the shear-wave number ($k_s = \omega / C_s$ with $C_s = \sqrt{m/r}$ the shear-wave velocity). The total field in the configuration is written as

$$u_y^t = u_y^f + u_y^s, \quad s_{xy}^t = s_{xy}^f + s_{xy}^s, \quad s_{zy}^t = s_{zy}^f + s_{zy}^s \quad (5)$$

in which u_y^f represents the free field, i.e. a given field in the absence of the crack, which u_y^s represented the scattered field. Both the total field as well as the free field has to satisfy Eq. (4). In addition to the differential equations, the field has to satisfy the stress-free boundary condition on the ground as well as the crack surfaces; however, on the crack surface the particle displacement jumps by a finite amount. Let

$$[u_y^t(\mathbf{x}_q)] = (u_y^t)_{\mathbf{x}_q}^+ - (u_y^t)_{\mathbf{x}_q}^- \quad (6)$$

in which $(u_y^t)_{\mathbf{x}_q}^+$ denotes the total displacement at the point Q (with position vector \mathbf{x}_q) lie on the

positive side of the crack which distinguished by a unit normal vector \mathbf{n}^+ pointed into the medium as shown in Fig. 1, and

$$\mathbf{n}^+ = \sin d \mathbf{e}_x + \cos d \mathbf{e}_z \quad (7)$$

The quantity $[u_y^t(\mathbf{x}_q)]$ can be regarded as the fundamental unknown. Moreover, the field has to satisfy the edge condition at the tips of the crack. This leads to (Karal and Karp, 1964)

$$[u_y^t] = 0(e^{1/2}), \quad \text{as } e \rightarrow 0 \quad (8)$$

in which e denotes the distance from some point in the half-space to a crack tip. As a final condition, one requires that u_y^s satisfies the radiation condition.

INTEGRAL EQUATION FOR THE JUMP IN THE PARTICLE DISPLACEMENT ACROSS THE CRACK

An integral representation for u_y^s can be derived by using the Betti-Rayleigh theorem (Kupradze, 1963). In this reciprocity relation two different elastodynamic states are employed. One of these states is taken to be Green's elastodynamic state, which is a solution of

$$\frac{\partial^2 \mathbf{u}^G}{\partial x^2} + \frac{\partial^2 u_y^G}{\partial z^2} + k_s^2 u_y^G + \frac{1}{m} d(\mathbf{x} - \mathbf{x}_p) = 0 \quad (9)$$

with B.C. $\mathbf{s}_{yz}^G|_{z=0} = 0$. Where d denotes the two-dimensional Dirac function and $\mathbf{x}_p = (x_p, z_p)$ denotes the position vector of a point P. By employing the scattered field as the other elastodynamic state in the Betti-Rayleigh theorem, once arrives at the integral representation

$$u_y^s(\mathbf{x}_p) = \int_{\Sigma} [u_y^t(\mathbf{x}_q)] \{ \mathbf{s}_{xy}^G(\mathbf{x}_q; \mathbf{x}_p) \mathbf{e}_x + \mathbf{s}_{yz}^G(\mathbf{x}_q; \mathbf{x}_p) \mathbf{e}_z \} \cdot \mathbf{n}^+ d\Sigma \quad (10)$$

in which the components \mathbf{s}_{xy}^G and \mathbf{s}_{yz}^G of the Green's tensor are consisted of free space part as well as the reflective homogeneous part and given by

$$\begin{aligned} \mathbf{s}_{xy}^G(x_q, z_q; x_p, z_p) &= \frac{1}{2p} \int_{-\infty}^{\infty} \left(\frac{-ik}{2n'} \right) e^{-n'|z_q - z_p| - ik(x_q - x_p)} dk + \frac{1}{2p} \int_{-\infty}^{\infty} \left(\frac{-ik}{2n'} \right) e^{-n'(z_q + z_p) - ik(x_q - x_p)} dk \\ \mathbf{s}_{yz}^G(x_q, z_q; x_p, z_p) &= \frac{1}{2p} \int_{-\infty}^{\infty} \left(\frac{-\text{sgn}(z_q - z_p)}{2} \right) e^{-n'|z_q - z_p| - ik(x_q - x_p)} dk + \frac{1}{2p} \int_{-\infty}^{\infty} \left(\frac{-1}{2} \right) e^{-n'(z_q + z_p) - ik(x_q - x_p)} dk \end{aligned} \quad (11)$$

where $n' = \sqrt{k^2 - k_s^2}$, $\text{Re} n' \geq 0$, one now substitutes expression Eq. (10) in the constitutive relation Eq. (3) and subsequently lets P approach Σ . Then using expression Eq. (5) and the traction free condition on the crack surface, one finally arrives at

$$-S_{yh}^f(\mathbf{x}_p, 0) = \frac{m}{2p} \int_{-a}^a [u_y^t(\mathbf{x}, 0)] I_k(\mathbf{x}, \mathbf{x}_p) d\mathbf{x}, \quad -a \leq x_p \leq a \quad (12)$$

where the kernel function $I_k(\mathbf{x}, \mathbf{x}_p)$ is expressed in spectral integral form as

$$I_k(\mathbf{x}, \mathbf{x}_p) = I_{k1}(\mathbf{x}, \mathbf{x}_p) + I_{k2}(\mathbf{x}, \mathbf{x}_p) \quad (13)$$

in which

$$I_{k1}(\mathbf{x}, \mathbf{x}_p) = \int_{-\infty}^{\infty} \{k^2 \sin^2 d - n'^2 \cos^2 d + ikn' \operatorname{sgn}[(x - x_p) \sin d] \sin(2d)\} / 2n' e^{-n'|x-x_p| \sin d - ik(x-x_p) \cos d} dk \quad (14)$$

$$I_{k2}(\mathbf{x}, \mathbf{x}_p) = \int_{-\infty}^{\infty} \{(k^2 - k_s^2 \cos^2 d) / 2n'\} e^{-n'[2d - (x+x_p) \sin d] - ik(x-x_p) \cos d} dk \quad (15)$$

In the derivation, we have made use of the following result

$$\begin{aligned} S_{yh}^s(x_p, z_p) &= m \left[\sin d \frac{\partial u_y^s}{\partial x_p} + \cos d \frac{\partial u_y^s}{\partial z_p} \right] \\ &= m \int_{\Sigma} [u_y^t(\mathbf{x}_q)] \left\{ \frac{\partial S_{xy}^G}{\partial x_p} \sin^2 d + \frac{\partial S_{yz}^G}{\partial z_p} \cos^2 d + \left(\frac{\partial S_{yz}^G}{\partial x_p} + \frac{\partial S_{xy}^G}{\partial z_p} \right) \sin d \cos d \right\} d\Sigma \end{aligned} \quad (16)$$

In view of the right hand side of Eq. (13), the first term $I_{k1}(\mathbf{x}, \mathbf{x}_p)$ represents the kernel function resulting from the free space part of the Green's tensor. Therefore, we can show that

$$I_{k1}(\mathbf{x}, \mathbf{x}_p) = I_{k1}^*(\mathbf{x}, \mathbf{x}_p) \equiv \int_{-\infty}^{\infty} (-n'/2) e^{-ik(x-x_p)} dk \quad (17)$$

where $I_{k1}^*(\mathbf{x}, \mathbf{x}_p)$ corresponds to the usual kernel function as the crack is in an infinite medium and consequently results into a hypersingular kernel as the source point coincides with the field point in which the integral over Σ exists in a sense of finite part. Whereas the second term $I_{k2}(\mathbf{x}, \mathbf{x}_p)$ in the right hand side of the Eq. (13) represents the kernel function resulting from the reflective homogeneous part of the Green's tensor, thus, it exhibits no any singularity even when the source point coincides with the field point.

METHOD OF SOLUTION

In order to solve the traction BIE (Eq. (12)), we discretize the crack Σ into N elements with equal length $2\Delta x_n$, and employ the central points x_n ($n=1 \sim N$) of each element as the collocation points, therefore, the integral over crack surface ($-a \leq x \leq a$) results into finite sum of N discrete element integrals each has an integration interval $x_n - \Delta x_n \leq x \leq x_n + \Delta x_n$. In addition, in order to take into account the edge condition at the crack tips (Eq. (8)), the distribution of the unknown boundary field $[u_y^t(\mathbf{x})]$ over an element is interpolated by a special shape function and normalized to the value at the collocation point (i.e. the central point) which is taken to be the nodal unknown.

$$[u_y^t(\mathbf{x})] = \frac{\sqrt{a^2 - \mathbf{x}^2}}{\sqrt{a^2 - \mathbf{x}_n^2}} [u_y^t(\mathbf{x}_n)] \quad ; \quad \mathbf{x}_n - \Delta \mathbf{x}_n \leq \mathbf{x} \leq \mathbf{x}_n + \Delta \mathbf{x}_n \quad (18)$$

In view of Eq. (18), the edge condition at the crack tips and the Hölder $C^{1,a}$ continuous, which ensures the existence of finite part at the collocation points, are fulfilled. Thus, the solution of the problem is deduced to solve the N nodal unknowns $[u_y^t(\mathbf{x}_n)]$, $n=1 \sim N$. By setting, in turn, the point \mathbf{x}_p in Eq. (12) chosen to be \mathbf{x}_m ($m=1 \sim N$), we obtain N algebraic equations just to solve the N nodal unknowns $[u_y^t(\mathbf{x}_n)]$

$$-S_{yh}^f(\mathbf{x}_m, 0) = \sum_{n=1}^N [K_{mn}(\mathbf{x}_m, \mathbf{x}_n) + L(\mathbf{x}_m, \mathbf{x}_n)] [u_y^t(\mathbf{x}_n)], \quad \forall m = 1 \sim N \quad (19)$$

where the matrices elements are defined by

$$K_{mn}(\mathbf{x}_m, \mathbf{x}_n) = \frac{m}{2p} \int_{\mathbf{x}_n - \Delta \mathbf{x}_n}^{\mathbf{x}_n + \Delta \mathbf{x}_n} \frac{\sqrt{a^2 - \mathbf{x}^2}}{\sqrt{a^2 - \mathbf{x}_n^2}} I_{k1}^*(\mathbf{x}, \mathbf{x}_m) d\mathbf{x} \quad (20)$$

$$L_{mn}(\mathbf{x}_m, \mathbf{x}_n) = \frac{m}{2p} \int_{\mathbf{x}_n - \Delta \mathbf{x}_n}^{\mathbf{x}_n + \Delta \mathbf{x}_n} \frac{\sqrt{a^2 - \mathbf{x}^2}}{\sqrt{a^2 - \mathbf{x}_n^2}} I_{k2}(\mathbf{x}, \mathbf{x}_m) d\mathbf{x} \quad (21)$$

When $\mathbf{x}_m \neq \mathbf{x}_n$ (i.e. off-diagonal element), the integration over \mathbf{x} can be evaluated by Gaussian quadrature and the wave number integral kernels I_{k1}^* or I_{k2} can be evaluated by modified steepest descent integration method (Yeh, *et al*, 2000), thus, we can obtain the matrices elements ($K_{mn} + L_{mn}$) without any difficulty. However, when $\mathbf{x}_m = \mathbf{x}_n$ (i.e. diagonal element), the integral $I_{k1}^*(\mathbf{x}, \mathbf{x}_n)$ in Eq. (20) diverges and becomes a hypersingular kernel, care must be taken to perform the integration on \mathbf{x} . In this case, we propose a regularization technique (subtracted an added back in spectral domain) to isolate the unbounded term and consequently extract the finite part from the hypersingular kernel function, for the details, the readers may refer to the report (Teng and Chai, 2002). Once the nodal unknowns are solved, the scattered surface displacement at point $\mathbf{x}_p = (x_p, 0)$ can be obtained according to Eq. (10), and the stress intensity factors K_3^\pm at crack tips are given by (Neerhoff and van der Hijden, 1984)

$$K_3^+ = \frac{m\sqrt{a}}{2} \frac{[u_y^t(\mathbf{x}_N)]}{\sqrt{a^2 - \mathbf{x}_N^2}}; \quad K_3^- = \frac{m\sqrt{a}}{2} \frac{[u_y^t(\mathbf{x}_1)]}{\sqrt{a^2 - \mathbf{x}_1^2}} \quad (22)$$

VERIFICATION OF NUMERICAL INPREMENTATION

In order to verify the proposed method of solution, we consider a crack with length $2a$ in infinite medium and subjected to an action of constant antiplane shear t_0 from which a rigorous exact solution is available (Knopoff, 1958)

$$[u_y^t(\mathbf{x})] = \frac{2t_0}{m} \sqrt{a^2 - x^2}; \quad |\mathbf{x}| \leq a, \quad S_{yh}^s(\mathbf{x}_p) = \begin{cases} -t_0; & |\mathbf{x}_p| \leq a \\ t_0(\mathbf{x}_p / \sqrt{\mathbf{x}_p^2 - a^2} - 1); & |\mathbf{x}_p| > a \end{cases} \quad (23)$$

In order to make against with the above solution, we consider the special case ($d/a \rightarrow \infty$, $d=0$) in Fig. 1, to simulate a horizontal crack in infinite medium. In addition, we adopt a plane SH wave normal incident to the crack and whose static limit as the excitation frequency tends to zero (i.e. $k_s \rightarrow 0$) corresponds to constant antiplane shear t_0 . Therefore, the corresponding free displacement and stress fields are given by

$$u_y^f(\mathbf{x}, h) = A e^{ik_s h} \quad ; \quad S_{yh}^f(\mathbf{x}, h) = m i k_s A e^{ik_s h} \quad (24)$$

where $A = m i k_s / t_0$. We take a normalized frequency $k_s a = 0.0001$ as zero frequency approximation, Figs. 2a and 2b show respectively the numerical results of the distribution of $[u_y^t(\mathbf{x})]$ along the crack surface and the scattered stress $S_{yh}^s(\mathbf{x}_p, 0)$ near the crack tips obtained from HBIEM and compared with those obtained from the static exact solution (Eq. (23)). From the figure, one may observe that both the zero frequency approximations as well as the validity of the proposed method are fulfilled with high accuracy.

SURFACE DEFORMATION OF A SHALLOW VERTICAL CRACK SUBJECTED TO A UNIFORM SHEAR STRESS DROP

In order to illustrate the whole proposed formulation, we consider a vertical crack (with length $2a$ and $d = p/2$) buried in the half-space with its center located at a depth d ($d > a$) below the ground surface and subjected to a horizontal standing plane SH wave whose static counterpart corresponds to a horizontal constant antiplane shear loading t_0 , the free field displacement and stress can be expressed as

$$u_y^f(\mathbf{x}, h) = A_h \sin(k_s h) \quad ; \quad S_{yh}^f(\mathbf{x}, h) = m k_s A_h \cos(k_s h) \quad (25)$$

from which the ground surface condition is satisfied and the required amplitude $A_h = t_0 / m k_s$, hence, at each collocation point \mathbf{x}_m ($-a \leq \mathbf{x}_m \leq a$, $h_m = 0$), the prescribed free field stress is $S_{yh}^f(\mathbf{x}_m, 0) = t_0$. We take a normalized frequency $k_s a = 0.0001$ as an approximation of static state. By employing the HBEM, we can calculate the relative slip $[u_y^t(\mathbf{x}_n)]$ at each collocation point from which the pseudo-static stress intensity factor K_3^\pm can be obtained under different d/a ratio, and the results are shown in Figs. 3a and 3b, respectively. From Fig. 3b, we can observe that the pseudo-static stress intensity factors K_3^\pm which obtained from HBEM are in excellent agreement with those obtained by static approach (Erdogan and Cook, 1974). Finally, employing the relative slip at each collocation points $[u_y^t(\mathbf{x}_n)]$, we can calculate the desired surface deformation $u_y^s(\mathbf{x}_p, 0)$ as shown in Fig. 3c. From Fig. 3, we can observe that the more profound of the results as the ratio d/a is more small, even though, the crack length and the prescribed shear stress keep fixed, therefore, the influence of the ground surface is significant for shallow fault system.

CONCLUSIONS

In this paper, we propose a dynamic hypersingular boundary element method (HBEM) to solve the surface deformation for constant stress drop slip zone (an antiplane crack) in elastic half-space in which zero frequency approximation is adopted to obtain the static counterpart. The proposed method (HBEM) is checked by comparing with other available data to show its validity of the method. The versatility of the method makes it possible to solve a more complicated problem, whenever, the static solution is difficult to obtain and the detail understanding of the surface deformation relative slip of cracked surface, and stress intensity factor are valuable in geophysics and earthquake engineering, for instance, if we know the relative slip of cracked surface, then we can calculate the seismic moment (Madariaga, 1979). Further, if we know about the stress intensity factor then the energy release rate can be calculated (Erdogan and Sih, 1963).

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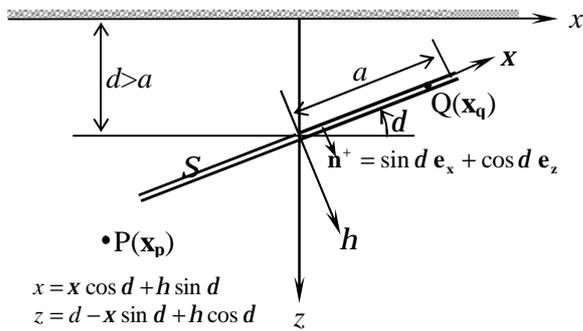


Fig. 1: Geometry of the crack Σ in the semi-infinite elastic medium.

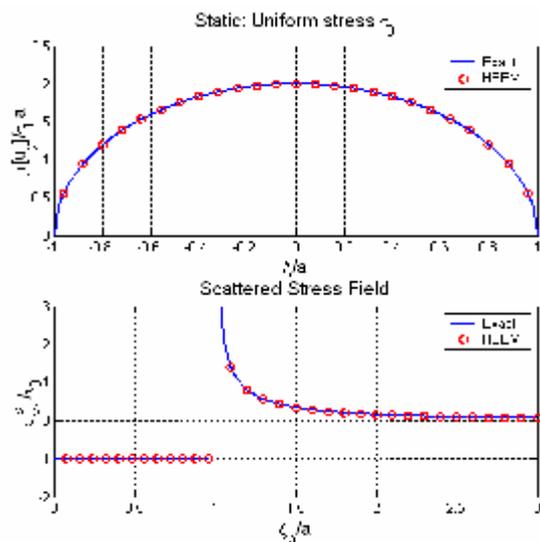
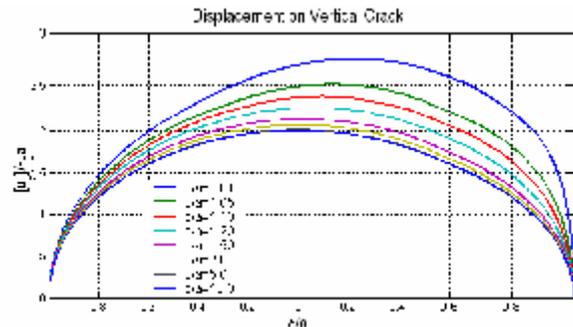
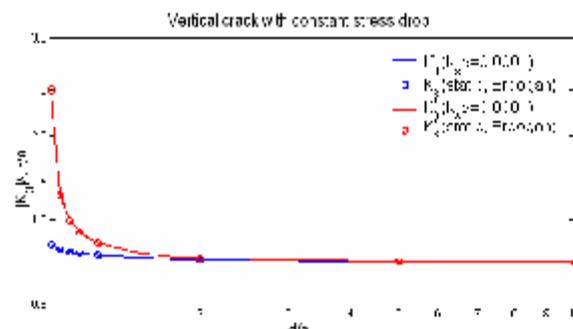


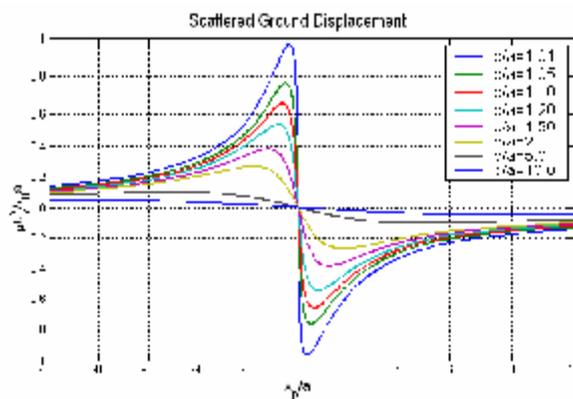
Fig. 2: The comparison of (a) $[u_y^t(x)]$ along the crack surface (b) $s_{yh}^s(x_p, 0)$ near the crack tip obtained from HBEM zero frequency approximation and from static exact solution.



(a)



(b)



(c)

Fig. 3: Numerical results of a vertical crack with different d/a ratio subjected to a prescribed uniform antiplane shear stress drop, (a) displacement on vertical crack, (b) stress intensity factor at crack tips and compared with static solution, (c) scattered ground displacement.